ON AN INTEGRAL EQUATION OF RADIATION – CONDUCTION HEAT TRANSFER

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We present a method for solving analytically the linearized integral equation of radiation–conduction heat transfer. Under certain conditions, realized in practice in heat conduction studies of translucent materials, a sufficiently precise solution may be obtained in closed form.

When the two mechanisms of energy transport, namely radiation and heat conduction, exist in a medium, the temperature field, as is well known, may be described by a nonlinear integro-differential equation [1-5]. Under certain conditions this equation can be linearized in such a way that the resulting mathematical simplification introduces negligible distortion in the temperature field. Realization of these conditions in practice occurs when the total temperature drop ΔT through a layer is significantly less than the absolute value of the general temperature background, T_0 [2, 6, 11]. Thus, in particular, an experiment can be arranged for studying the thermophysical characteristics of translucent materials in which radiation–conduction heat transfer is present. The linearized equation is a Fredholm integral equation of the second kind, so that the questions as to its solvability and the uniqueness of a solution are answerable through the general theory and require no special consideration. We present below an effective method for obtaining this solution. So far as is known to us, a practical solution in analytical form of the equation of radiation–conduction heat transfer has not been obtained up to the present time: either the form of the function describing the temperature field was assumed (as, for example, in [2, 5, 6]), effectively eliminating thereby any need to solve the equation, replacing the latter instead by the evaluation of certain parameters present in the function assumed; or the solution was obtained numerically on an electronic digital computer for separate special cases [4, 8].

We present here a method for studying the simplest case, namely, that of radiation–conduction in a planar wall, with absolutely black boundaries and without heat sources, when the linearized equation has the form [11]

$$\varphi(\tau) = f(\tau) - N\int_0^{\tau_0} E_3|\tau-\tau'|\,\varphi(\tau')\,d\tau', \tag{1}$$

where

$$f(\tau) \doteq aN\tau - N\frac{\Delta T}{T_0}E_3(\tau_0-\tau) + C;$$

C is a constant determined by the boundary condition $\varphi(0) = 0$. Consideration of the more general case, involving reflecting boundary walls, entails no major changes; it merely increases the volume of calculations associated with the more complicated free function of Eq. (1) and its kernel, the latter depending now on both the difference and the sum of its arguments [11]. When internal heat sources are present, with temperature-dependent strengths, only the form of $f(\tau)$ is affected.

Since the kernel of Eq. (1) does not possess sufficient smoothness, a known procedure for approximating the solution of such equations, consisting of replacing the kernel by a degenerate kernel by representing it by a section of a Fourier series, leads to a linear algebraic system of higher order [9]. This circumstance does not permit us to obtain the solution in analytical form in the usual way, which, in a number of cases, is very necessary.

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A First Approximation

We study the nature of the Fourier coefficients for the kernel of the equation, considered as a function of a single variable varying over the interval (–1, 1). Since the kernel is symmetric we may select as our orthonormal system a trigonometric system of cosines; we then have

$$E_3|\tau-\tau'| = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi|\tau-\tau'|}{\tau_0}, \tag{2}$$

where

$$A_0 = \frac{1}{\tau_0}[E_4(0) - E_4(\tau_0)];$$
$$A_n = \frac{2\tau_0}{n^2\pi^2}[E_2(0) - (-1)^n E_2(\tau_0)] - \frac{2\tau_0^2}{n^3\pi^3}\left[\operatorname{arctg}\frac{n\pi}{\tau_0} - \operatorname{Im} E_1(\tau_0 + in\pi)\right]. \tag{3}$$

Further, following the usual procedure for improving the convergence of a series, we sum out the slowly converging part of the series in Eq. (2), the part associated with the jumps of the function and its derivatives. In addition, we employ the known relations [10]

$$\sum_{k=1}^{\infty}\frac{\cos k\pi x}{k^{2n}} = (-1)^n\frac{1}{2}\frac{(2\pi)^{2n}}{(2n)!}B_{2n}\left(\frac{x}{2}\right);$$
$$\sum_{k=1}^{\infty}\frac{\sin k\pi x}{k^{2n+1}} = (-1)^n\frac{1}{2}\frac{(2\pi)^{2n+1}}{(2n+1)!}B_{2n+1}\left(\frac{x}{2}\right) \quad (0<x<1); \tag{4}$$
$$\sum_{k=1}^{\infty}(-1)^k\frac{\cos kx}{k^{2q}} = P_{2q}(x); \quad \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kx}{k^{2q+1}} = P_{2q+1}(x) \quad (-\pi\leqslant x\leqslant\pi).$$

Here $B_m(x)$ are the Bernoulli polynomials and the $P_n(x)$ are polynomials compatible with them, defined through the recursion relation

$$P_n'(x) = (-1)^n P_{n-1}(x); \quad P_1(x) = \frac{x}{2}.$$

Then we obtain

$$E_3|\tau-\tau'| = k(\tau, \tau') + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi|\tau-\tau'|}{\tau_0}, \tag{5}$$

where

$$k(\tau, \tau') = M_1 + M_2\tau\tau' + M_3(\tau^2 + \tau'^2) - |\tau-\tau'|;$$
$$M_1 = \frac{1}{\tau_0}[E_4(0) - E_4(\tau_0)] + \frac{\tau_0}{3} + \frac{\tau_0 E_2(\tau_0)}{6};$$
$$M_2 = -\frac{1}{\tau_0}[1 - E_2(\tau_0)] = -2M_3; \tag{6}$$
$$B_n = -\frac{2\tau_0^2}{n^3\pi^3}\left[\operatorname{arctg}\frac{n\pi}{\tau_0} - \operatorname{Im} E_1(\tau_0 + in\pi)\right].$$

Thus the kernel of Eq. (1) is represented in the form of a sum of an algebraic function and a trigonometric kernel, the coefficients of which decrease fairly rapidly. We note that for small values of τ_0 (which corresponds to small thicknesses or high transparency of the material), the second term in Eq. (5) may be discarded and $k(\tau, \tau')$ then serves as a good approximation for the kernel of Eq. (1). In Table 1 we show the values of the error Δ due to such an approximation for two values of τ_0.

If we replace $E_3|\tau-\tau'|$ by $k(\tau, \tau')$, we obtain, in place of Eq. (1), the new integral equation (at the same time, we have carried out the change of variables $x = \tau - l$; $y = \tau' - l$)

$$\varphi_1(x) = f(x) - N\int_{-l}^{l} k(x, y)\varphi_1(y)\,dy, \tag{7}$$

TABLE 1. Values of the Error Δ for Two Values of τ_0

τ_0	$\{E_3\lvert\tau-\tau'\rvert\}_{max}$	$\{E_3\lvert\tau-\tau'\rvert\}_{min}$	Δ
0,20	0,500	0,352	$\leqslant 0{,}5\cdot 10^{-2}$
0,55	0,500	0,206	$\leqslant 3{,}6\cdot 10^{-2}$

the solution of which may be obtained in closed form and is a first approximation to the solution of Eq. (1). Differentiating Eq. (7) three times, we have

$$\varphi_1''' = f''' + 2N\varphi_1' \tag{8}$$

and then

$$\varphi_1(x) = \frac{C_1}{m} e^{mx} - \frac{C_2}{m} e^{-mx} + C_3 + f + \frac{m^2}{2}[U_1(x) + U_2(x)], \tag{9}$$

where

$$U_1(x) = \int e^{mx}\left(\int f e^{-mx} dx\right) dx; \quad U_2(x) = \int e^{-mx}\left(\int f e^{mx} dx\right) dx; \quad m^2 = 2N.$$

Substituting Eq. (9) into Eq. (7) and equating coefficients having the same powers of x, we obtain the following three equations for determining the constants C_1, C_2, and C_3:

$$\begin{aligned}
&-M_3 \operatorname{sh} ml C_1 + M_3 \operatorname{sh} ml C_2 + m^2\left(\frac{1}{2} - M_3 l\right) C_3 = NM_3[V_1(l) - V_1(-l)];\\
&\left[\frac{M_2}{m}\operatorname{sh} ml - (M_2 l + 1)\operatorname{ch} ml\right] C_1 + \left[\frac{M_2}{m}\operatorname{sh} ml - (M_2 l + 1)\operatorname{ch} ml\right] C_2\\
&\qquad = -aN + NM_2[V_2(l) - V_2(-l)] + N[V_1(l) + V_1(-l)];\\
&\left\{\left[l - M_1 - M_3\left(l^2 + \frac{2}{m^2}\right)\right]\operatorname{sh} ml - \frac{\operatorname{ch} ml}{m}(1 - 2M_3 l)\right\} C_1\\
&\quad -\left\{\left[l - M_1 - M_3\left(l^2 + \frac{2}{m^2}\right)\right]\operatorname{sh} ml - \frac{\operatorname{ch} ml}{m}(1 - 2M_3 l)\right\} C_2\\
&\quad + \left[-1 + m^2\left(\frac{l^2}{2} - M_1 l - \frac{M_3 l^3}{3}\right)\right] C_3 = NM_1[V_1(l) - V_1(-l)]\\
&\qquad - N[V_2(l) + V_2(-l)] + NM_3[V_3(l) - V_3(-l)].
\end{aligned} \tag{10}$$

Here we have introduced the notation

$$V_i(x) = \int x^{i-1}\left[f(x) + \frac{m^2}{2}(U_1 + U_2)\right] dx; \quad i = 1;\ 2;\ 3.$$

After completing the calculations, we obtain

$$\begin{aligned}
\varphi_1(\tau) = {} & \frac{C_1}{m} e^{-ml}(e^{m\tau} - 1) - \frac{C_2}{m} e^{ml}(e^{-m\tau} - 1) + \frac{\Delta T}{2T_0}[E_2(\tau_0 - \tau) - E_2(\tau_0)]\\
& - \frac{\Delta T}{4mT_0}\{e^{-m(\tau_0-\tau)} E_1[(1 - m)(\tau_0 - \tau)] - e^{m(\tau_0-\tau)} E_1[(1 + m)(\tau_0 - \tau)]\\
& - e^{-m\tau_0} E_1[(1 - m)\tau_0] + e^{m\tau_0} E_1[(1 + m)\tau_0]\}.
\end{aligned} \tag{11}$$

From Eqs. (10) we can obtain explicit expressions for C_1 and C_2, which, in view of their complexity, will not be given here (see [11]). If in Eq. (11) we put $\tau = \tau_0$, we obtain an equation for determining the total drop ΔT through a layer.

Using Bateman's method [9] to construct the successive corrections (taking $k(\tau, \tau')$ as the auxiliary kernel in this method), we can estimate the error of the first approximation. It turns out that in a number of cases $\varphi_1(\tau)$ is very close to the exact solution of Eq. (1). Thus, for example, for $\tau_0 = 0.138$ and N = 24.8, the error of $\varphi_1(\tau)$ does not exceed the quantity $6.7\cdot 10^{-3}\ \|f\|$.

Determination of the Characteristic Values of Eq. (1)

This can also be accomplished by using the approximated kernel. To do this it is sufficient to put $f(x) \equiv 0$ and equate the determinant of the system (10) to zero; this leads to the two equations:

$$\operatorname{th} z = z\left(1 + \frac{1}{M_2 l}\right), \tag{12}$$

$$\operatorname{th} z = \frac{(1-2M_3 l)^2 z}{z^2\left(1+\frac{4}{3}M_3^2 l^2 - \frac{M_1}{l} - 2M_3 l\right) - 4lM_3(1-M_3 l)}, \tag{13}$$

the roots z_n of which are connected with the approximate values of the characteristic numbers of the kernel $-E_3|\tau-\tau'|$ by the equations

$$\tilde{N}_n = \frac{2}{\tau_0^2} z_n^2. \tag{14}$$

It is readily seen that Eq. (12) has z = 0 as its only root, so that we need consider only Eq. (13). The corresponding approximate expressions for the characteristic functions may be obtained from Eq. (9). They have the form

$$\tilde{\varphi}_n(\tau) = -\left[\operatorname{ch} m_n\tau + \frac{M_3 \operatorname{sh} m_n l}{\tilde{N}_n(1-2M_3 l)}\right]. \tag{15}$$

We clarify now just how close the values $\tilde{N}_n$ so obtained are to the true critical values of the parameter N. For this purpose we write the integral operator of Eq. (1) in the form of a sum $K\varphi = K_1\varphi + K_2\varphi$, where K_1 and K_2 are operators with kernels corresponding to the first and second terms in Eq. (5). As is well known [7], the characteristic numbers $\mu_n = 1/N_n$ of the operator K and $\bar{\mu}_n = 1/N_n$ of the operator K_1 are not separated from one another by more than $\|K_2\|$. We have

$$\|K_2\|^2 \leqslant \int_0^{\tau_0}\int_0^{\tau_0}\left\{\sum_{n=1}^{\infty} B_n \cos\frac{n\pi(x-y)}{\tau_0}\right\}^2 dxdy = \int_0^{\tau_0}\int_0^{\tau_0}\left\{\sum_{n,m=1}^{\infty} B_n B_m \cos\frac{n\pi(x-y)}{\tau_0}\cos\frac{m\pi(x-y)}{\tau_0}\right\} dxdy$$

and by virtue of the uniform convergence of the last series we find, after making the calculations, that

$$\|K_2\|^2 \leqslant \tau_0^2\left(\frac{B_1^2}{2} + \frac{40}{9\pi^2}B_1B_2 + \frac{B_2^2}{2} + \frac{104}{25\pi^2}B_2B_3 + \frac{136}{225\pi^2}B_1B_4 + \cdots\right). \tag{16}$$

The series in the right member of inequality (16) converges no worse than $1/n^6$; thus this inequality enables us to obtain the necessary estimate. Calculations made for $\tau_0 = 0.2$ and $\tau_0 = 0.55$ show that the maximum characteristic values in these two cases amount to -0.119 and -0.324, the errors made in determining them, to a first approximation, being no greater than 0.5% and 3.9%, respectively.

If the optical thickness τ_0 of a layer is of the order of unity or larger, the first approximation turns out to be very crude, a fact which reflects the poor approximation of the kernel of the equation by means of the function $k(\tau, \tau')$. In this case it behoves us to retain the first few terms of the series in Eq. (5), their number being, however, comparatively small (the coefficients B_n, as is evident, decrease as $1/n^3$). The largest error of the approximation corresponds to the values $\tau = \tau'$; therefore the required number of terms of the series in Eq. (5) can be determined by equating the sum $k(0) + B_1 + B_2 + \ldots + B_p$ to $E_3(0) = 1/2$. Substituting the new expression for the kernel into Eq. (1) and differentiating three times as before, we obtain a differential equation whose solution yields a more precise approximation to the solution of the original equation:

$$\varphi(\tau) = \varphi_1(\tau) + \sum_{k=1}^{p}\frac{\tau_0}{k\pi}\left[H_k\left(\sin\frac{k\pi(\tau-l)}{\tau_0} + \sin\frac{k\pi}{2}\right) - F_k\left(\cos\frac{k\pi(\tau-l)}{\tau_0} - \cos\frac{k\pi}{2}\right)\right]. \tag{17}$$

The coefficients C_i, H_k, and F_k may be obtained from the system of equations

$$-M_3\operatorname{sh} ml C_1 + M_3 \operatorname{sh} ml C_2 + N(1-2M_3 l)C_3 + NM_3\sum_{k=1}^{p}F_k\frac{2\tau_0^2}{k^2\pi^2}\sin\frac{k\pi}{2} = NM_3[V_1(l) - V_1(-l)];$$

$$\left[-(1+M_2 l)\operatorname{ch} ml + M_2\frac{\operatorname{sh} ml}{m}\right]C_1 + \left[-(1+M_2 l)\operatorname{ch} ml + M_2\frac{\operatorname{sh} ml}{m}\right]C_2$$

$$-N\sum_{k=1}^{p}H_k\frac{\tau_0}{k\pi}\left[M_2\left(\frac{2\tau_0^2}{k^2\pi^2}\sin\frac{k\pi}{2} - \frac{\tau_0^2}{k\pi}\cos\frac{k\pi}{2}\right) - \frac{2\tau_0}{k\pi}\cos\frac{k\pi}{2}\right]$$

$$= -aN + NM_2[V_2(l) - V_2(-l)] + N[V_1(l) + V_1(-l)];$$

$$\left\{-M_1\operatorname{sh} ml - M_3\left[\left(l^2+\frac{2}{m^2}\right)\operatorname{sh} ml - \frac{2l}{m}\operatorname{ch} ml\right] + l\operatorname{sh} ml - \frac{\operatorname{ch} ml}{m}\right\}C_1$$

$$+\left\{M_1 \operatorname{sh} ml + M_3\left[\left(l^2+\frac{2}{m^2}\right)\operatorname{sh} ml - \frac{2l}{m}\operatorname{ch} ml\right] - l \operatorname{sh} ml + \frac{\operatorname{ch} ml}{m}\right\} C_2$$

$$+\left[-1 - m^2\left(M_1 l + \frac{M_3 l^3}{3} - \frac{l^2}{2}\right)\right] C_3 + N\sum_{k=1}^{p} F_k \frac{\tau_0}{k\pi}\left[M_1 \frac{2\tau_0}{k\pi}\sin\frac{k\pi}{2}\right.$$

$$+ M_3\left(\frac{\tau_0^3}{2k\pi}\sin\frac{k\pi}{2} + \frac{2\tau_0^3}{k^2\pi^2}\cos\frac{k\pi}{2} - \frac{4\tau_0^3}{k^3\pi^3}\sin\frac{k\pi}{2}\right) - \frac{\tau_0^2}{k\pi}\sin\frac{k\pi}{2}$$

$$\left. + 2\frac{\tau_0^2}{k\pi}\cos\frac{k\pi}{2}\right] = NM_1[V_1(l) - V_1(-l)] + NM_3[V_3(l) - V_3(-l)] - N[V_2(l) + V_2(-l)]; \tag{18}$$

$$S_{i1}C_1 + S_{i2}C_2 + S_{i3}C_3 + \sum_{\substack{k=1\\k\neq i}}^{p} S_{ik}F_k + S_{ii}F_i = NB_i[W_i^{(1)}(l) - W_i^{(1)}(-l)];$$

$$S_{j1}C_1 + S_{j2}C_2 + S_{j3}C_3 + \sum_{\substack{k=1\\k\neq j}}^{p} S_{jk}H_k + S_{jj}H_j = NB_j[W_j^{(2)}(l) - W_j^{(2)}(-l)]$$

$$(i,\ j = 1,\ 2,\ \ldots\ p).$$

Here

$$W_n^{(1)}(x) = \int \cos\frac{n\pi x}{\tau_0}\left[f(x) + \frac{m^2}{2}(U_1 + U_2)\right]dx;$$

$$W_n^{(2)}(x) = \int \sin\frac{n\pi x}{\tau_0}\left[f(x) + \frac{m^2}{2}(U_1 + U_2)\right]dx;$$

$$S_{i1} = \frac{2NB_i}{m\left[m^2 + \left(\frac{i\pi}{\tau_0}\right)^2\right]}\left(m \operatorname{sh} ml \cos\frac{i\pi}{2} + \frac{i\pi}{\tau_0}\operatorname{ch} ml \sin\frac{i\pi}{2}\right) = S_{i2};$$

$$S_{i3} = -2NB_i\frac{\tau_0}{i\pi}\sin\frac{i\pi}{2};$$

$$S_{ik} = NB_k\frac{\tau_0^2}{k\pi^2}\left[\frac{1}{i+k}\sin\frac{(i+k)\pi}{2} + \frac{1}{i-k}\sin\frac{(i-k)\pi}{2}\right];$$

$$S_{j1} = -\frac{2NB_j}{m\left[m^2 + \left(\frac{j\pi}{\tau_0}\right)^2\right]}\left(m\sin\frac{j\pi}{2}\operatorname{ch} ml - \frac{j\pi}{\tau_0}\cos\frac{j\pi}{2}\operatorname{sh} ml\right) = S_{j2};$$

$$S_{jk} = NB_k\frac{\tau_0^2}{k\pi^2}\left[\frac{1}{j+k}\sin\frac{(j+k)\pi}{2} - \frac{1}{j-k}\sin\frac{(j-k)\pi}{2}\right];$$

$$S_{ii} = \frac{\tau_0}{i\pi} - 2N\left(\frac{\tau_0}{i\pi}\right)^3;$$

$$S_{jj} = -\frac{\tau_0}{j\pi} - 2N\left(\frac{\tau_0}{j\pi}\right)^3.$$

Calculations carried out for $\tau_0 = 1.88$ show that when a total of two terms of Eq. (5) is taken, we obtain the value of $E_3|\tau - \tau'|$ with an error no greater than 8%. In this case the system (18) consists of seven equations, so that determining all the coefficients presents no problems.

NOTATION

ΔT	is the temperature drop in the layer;
T_0	is the temperature of the surface at the heater;
λ	is the thermal conductivity;
k	is the absorption index,
σ	is the Stefan constant;
n	is the refraction index;
Q	is the total energy flux through the layer;
h	is the thickness of the layer;
$\tau_0 = kh$	is the optical thickness;
$\tau = kx$ and $\tau' = k\xi$	are the dimensionless coordinates;

$\varphi(\tau) = (T_0 - T(\tau))/T_0$ is the dimensionless relative temperature;

$E_n(x) = \int_1^{\infty} (e^{-xt}/t^n)\,dt$ are the integro-exponential functions;

$N = 8n^2 \sigma T_0^3 / k\lambda$;

$a = Q/8n^2 \sigma T_0^4$.

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